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A critique of the kinematic structure of the Rarita-Schwinger framework

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After a brief review of the celebrated 1941 paper of Rarita and Schwinger on the theory of particles with half-integral spins, we present an *ab initio* construct of the representation space relevant for the description of spin-$\frac{3}{2}$ particles. The chosen example case of spin-$\frac{3}{2}$ shows that covariance of a wave equation, and those of the imposed supplementary conditions, alone is not a sufficient criterion for establishing the compatibility of a framework with relativity – a lesson already arrived by Velo and Zwanziger. Here this same result is shown to hold purely at the level of the representation space without invoking any interactions. The detailed analysis forces us to abandon a single-spin interpretation of the Rarita and Schwinger framework, and suggests a new interpretation that fully respects the relativity theory.

1 Introduction

Based on the analysis of then-existing data, Oppenheimer put forward the suggestion that electron neutrino may carry a mass, and that it may have a spin of three half [1]. While current experiments indicate that neutrinos carry a non-zero mass,[1] the question on the spin of electron neutrino was immediately settled by a set of two papers. Rarita and Schwinger [7] provided a brief one-and-a-half column letter in *Physical Review* on the theory of particles with half-integral spins, while in an accompanying letter Kusaka ruled out the possibility of neutrino with spin $\frac{3}{2}$.

[1] More precisely, what the current experiments are probing is the phenomena of flavor oscillations. In such a scenario a flavor eigenstate is a linear superposition of different mass eigenstates[2–6].

This month (July 2001) falls on the sixtieth anniversary of Ref. [7]. The interest in Rarita-Schwinger formalism remains unabated as more and more baryonic resonances of higher spins are found in particle detectors on the one hand, and as theorists realize that for one reason, or another, higher spins may play a pivotal role in the unification of gravity with other interactions. Yet, this sixty-year old formalism remains vexing to theorists to some extent. This circumstance arises due to difficulties with the quantization of this field on the one hand, and tachyonic propagation on the other. We conjecture that all the known difficulties associated with the Rarita-Schwinger formalism carry their origins from the improper treatment and interpretation of the underlying representation space. In conjunction with Ref. [11], this paper is a preliminary step towards exploring this conjecture.

Here we first retrace the arguments of Rarita and Schwinger, and then immediately proceed to construct the representation space defined by Eq. (1), below. This would allow us to present an essentially self-contained completion of the Rarita-Schwinger framework that is consistent with the relativity theory at the kinematic level. At the same time it will allow us to point out where and how the inconsistency in the canonical wisdom on the Rarita-Schwinger framework enters.

Our considerations will be confined to spin-$\frac{3}{2}$. No new conceptual difficulties are expected to enter for spins $s > \frac{3}{2}$.

2 Rarita-Schwinger framework for spin-$\frac{3}{2}$

The Rarita and Schwinger spinor-vector, ψ_μ, transforms as a finite dimensional non-unitary representation of the Lorentz group:

$$\underbrace{\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]}_{\text{SPINOR SECTOR}}\otimes\underbrace{\left(\frac{1}{2},\frac{1}{2}\right)}_{\text{VECTOR SECTOR}} \tag{1}$$

It satisfies the wave equation:

$$\left(i\gamma^\lambda\partial_\lambda - m\right)\psi_\mu = 0 \tag{2}$$

The ψ_μ, as is evident from Eq. (1), contains 16 degrees of freedom. In the original interpretation, those were (correctly)interpreted to be distributed over two Dirac spinors, $\partial^\mu\psi_\mu$ and $\gamma^\mu\psi_\mu$, and the eight degrees of freedom required for the description of a spin-$\frac{3}{2}$ particle and its antiparticle. The idea was put forward to nullify the indicated Dirac spinors in the hope that in this way they will be removed from the representation space. In doing so one eventually would end up with eight degrees of freedom as required for the description of a spin-$\frac{3}{2}$ field.

That, $\psi_I \equiv \partial^\mu \psi_\mu$, and, $\psi_{II} \equiv \gamma^\mu \psi_\mu$, indeed satisfy the Dirac equation is immediately seen from the following two simple exercises:

I. Taking the divergence of Eq. (2) shows that ψ_I does indeed satisfy the Dirac equation:

$$\partial^\mu (i\gamma^\lambda \partial_\lambda - m)\psi_\mu = \left(i\gamma^\lambda \partial_\lambda - m\right) \partial^\mu \psi_\mu = 0\,. \tag{3}$$

That is,

$$(i\gamma^\lambda \partial_\lambda - m)\psi_I = 0 \tag{4}$$

The situation is slightly trickier, but not fatally, with ψ_{II}.

The Rarita-Schwinger framework sets, $\psi_I = 0$, and also, $\psi_{II} = 0$.

II. In nullifying ψ_I, i.e., in setting $\partial^\mu \psi_\mu = 0$, allows for $\gamma^\mu \psi_\mu$ to satisfy a Dirac equation (with the wrong sign for the mass term):

$$\begin{aligned}(i\gamma^\lambda \partial_\lambda - m)\gamma^\mu \psi_\mu &= (i\partial_\lambda \gamma^\lambda \gamma^\mu - m\gamma^\mu)\psi_\mu \\ &= (2i\partial_\lambda g^{\lambda\mu} - i\partial_\lambda \gamma^\mu \gamma^\lambda - m\gamma^\mu)\psi_\mu \\ &= 2i\partial^\mu \psi_\mu - \gamma^\mu (i\partial_\lambda \gamma^\lambda + m)\psi_\mu \end{aligned} \tag{5}$$

Now, $\partial^\mu \psi_\mu$ equals ψ_I, which the Rarita-Schwinger framework sets equal to zero. The second term on the right-hand side of the above equation carries a wrong sign for the mass term (if ψ_{II} is to satisfy the Dirac equation). This, however, can be corrected by replacing the original suggestion of ψ_{II} by $\psi'_{II} \equiv \gamma^5 \psi_{II}$. Then, it is clear that ψ'_{II} satisfies the Dirac equation. This is an important point as regards the relative intrinsic parities of the spin-$\frac{1}{2}$ particles contained in the representation space (1). However, for the rest of this paper we shall ignore this "minor" matter of inconsistency in the Rarita-Schwinger framework. The reason we ignore this matter is justified on the grounds that it does not affect our essential conclusions in any way. However, the reader should keep the presence of γ^5 in mind while applying the framework to physical problems.

In summary, the Rarita-Schwinger framework for spin-$\frac{3}{2}$ consists of Eq. (2), supplemented by conditions:

$$\gamma^\mu \psi_\mu = 0\,, \tag{6}$$
$$\partial^\mu \psi_\mu = 0\,. \tag{7}$$

This framework is then claimed to describe a pure spin-$\frac{3}{2}$ system, despite a parenthetic remark in the original paper of Rarita and Schwinger which read, "it [the square of the intrinsic angular momentum] will not have this value $[\frac{3}{2}]$ in an arbitrary reference frame." While our analysis will explicitly support this remark, we will show that despite covariance of the system of Eqs. (2), (6), and (7), the Rarita-Schwinger framework is incompatible with the theory of relativity.

3 Kinematic structure of the Rarita-Schwinger framework

The most noted problems with the above summarized framework have been given by Johnson and Sudarshan [9], on one hand, and by Velo and Zwanziger [10] on the other. These authors studied propagation of Rarita-Schwinger field in an external electromagnetic potential, and the latter authors came to the conclusion that "the main lesson to be drawn ... is that special relativity is not automatically satisfied by writing equations that transform covariantly. In addition, the solutions must not propagate faster than light."

Here, essentially the same result is shown to hold purely at the level of the representation space without invoking any interactions — provided that one takes due care, beyond the work of Refs. [9,10], of the $\left(\frac{1}{2},\frac{1}{2}\right)$ sector of the theory. The detailed analysis presented in here forces us to abandon a single-spin interpretation of the Rarita and Schwinger framework, and suggests a new interpretation that fully respects the relativity theory — at least at the kinematic level.[2] The new interpretation of the representation space defined by Eq. (1) will require us to abandon the supplementary conditions, (7) and (6), and force us to interpret this space as a multi-spin object containing two spin half objects of opposite relative intrinsic parities, and a spin three-half object.

3.1 Incompatibility of the Rarita-Schwinger framework with theory of relativity

The un-truncated Rarita-Schwinger representation space is a direct product of a spinor and a Lorentz vector. The objects which span the spinor and vector

[2] We do not hasten to study as to what happens when interactions are introduced. The reason is simple: if the kinematic structure itself is acausal, or pathological in any manner, then these same elements would come to plague us later when interactions are introduced. In particular, we draw attention to Eq. (16) of Ref. [11] which indicates as to what could have gone wrong even with the completeness relation for the $\left(\frac{1}{2},\frac{1}{2}\right)$ sector of the theory. For any theory that does not satisfy the correct completeness relation, quatization is bound to be problematic.

sectors of the theory are obtained by applying the $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ boost to the following rest spinors:

$$\psi_1(\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \psi_2(\vec{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \psi_3(\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \psi_4(\vec{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \tag{8}$$

and the $\left(\frac{1}{2},\frac{1}{2}\right)$ boost, to the following Lorentz vectors in the rest frame:

$$w_1(\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, w_2(\vec{0}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, w_3(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, w_4(\vec{0}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \tag{9}$$

The $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ and the $\left(\frac{1}{2},\frac{1}{2}\right)$ boosts are:

$$\kappa^{\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)} = \kappa^{\left(\frac{1}{2},0\right)} \oplus \kappa^{\left(0,\frac{1}{2}\right)}, \quad \kappa^{\left(\frac{1}{2},\frac{1}{2}\right)} = \kappa^{\left(\frac{1}{2},0\right)} \otimes \kappa^{\left(0,\frac{1}{2}\right)}, \tag{10}$$

with

$$\kappa^{\left(\frac{1}{2},0\right)} = \frac{1}{\sqrt{2m(E+m)}}\left[(E+m)I_2 + \vec{\sigma}\cdot\vec{p}\right] \tag{11}$$

$$\kappa^{\left(0,\frac{1}{2}\right)} = \frac{1}{\sqrt{2m(E+m)}}\left[(E+m)I_2 - \vec{\sigma}\cdot\vec{p}\right] \tag{12}$$

All notational details are those of Ref. [11].

A careful reader has perhaps already noted that the application of the boost operators takes one from the original laboratory frame to a boosted frame. However, as no inertial frame is a preferred frame, the boosted objects should also exist in the original frame. It is by this "Wigner argument" that the laboratory frame is populated with spinors and vectors for all values of $\vec{p}$.

The $\psi_i(\vec{p})$, $i = 1,2,3,4$, carry well-defined spin, i.e. $s = \frac{1}{2}$, while the same is **not** true for $w_\zeta(\vec{p})$, $\zeta = 1,2,3,4$. However, for $\vec{p} = \vec{0}$, the latter, for $\zeta = 1,2,3$ are eigenstates of spin one, while the $\zeta = 4$ case yields spin zero. The interested reader will find that this result is in accord with observation of Rarita and Schwinger in the context of spin three half — see, the parenthetic remark after Eq. (2) of Ref. [7]. Further details on the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation space can be found in Ref. [11].

After rotation by the matrix,

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & -i & 0 \\ -i & 0 & 0 & i \\ 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \end{pmatrix}, \tag{13}$$

obtained in Ref. [11], the $w_\zeta(\vec{p})$, carry the usual (contravariant)Lorentz index. We represent this S-rotated object by $[W_\zeta(\vec{p})]^\mu$. In this language (and in momentum space), the 16 objects that span the representation space defined in eq. (1) are obtained as:

$$\psi^\mu_{i\zeta}(\vec{p}) \equiv \psi_i(\vec{p}) \otimes [W_\zeta(\vec{p})]^\mu\,(\vec{p}) \tag{14}$$

In order that $\gamma_\mu \psi^\mu_{i\zeta}$ identically vanish for *all* values of i and ζ, we find that:

$$(E+m)^2 - \vec{p}^{\,2} = 0. \tag{15}$$

Solving for E, this gives

$$E = -m \pm \sqrt{\vec{p}^{\,2}}. \tag{16}$$

As such, the group velocity associated with the Rarita-Schwinger field turns out to be:

$$\vec{v}_g \equiv \frac{\partial E}{\partial \vec{p}} = 1\,\widehat{p} \tag{17}$$

That is, implementing the supplementary condition, Eq. (6), requires the group velocity associated with the Rarita Schwinger field to be unity (i.e. velocity of light). This value of unity is *independent of mass.* Consequently, we conclude that the covariance of a set of equations alone is not sufficient to warrant consistency with the theory of relativity. One must further demand that one obtains the correct dispersion relation.

The supplementary condition (6) involves not only a summation over the Lorentz indices, but also involves a transformation on the relevant spinorial elements. In contrast, the supplementary condition (7) sums out the Lorentz index, and without any further transformation on the spinorial element sets it equal to zero. It is therefore instructive to look at the Lorentz-index defining $\left(\frac{1}{2},\frac{1}{2}\right)$ representation space, to gain further insight in the representation space (1).

A direct calculation of the divergency of each one of the four Lorentz vectors $[W_\zeta]^\mu$, $\zeta = 1,2,3,4$, leads to

$$\begin{aligned} p_\mu\,[W_\zeta]^\mu &= c_\zeta(m^2 - p^2) = 0 \quad \text{for} \quad \zeta = 1,2,3\,, \\ p_\mu\,(W_4)^\mu &= \frac{i}{m}p^2\,. \end{aligned} \tag{18}$$

Here, $c_1 = i(p_x + ip_y)$, $c_2 = -ip_z$, and $c_3 \doteq -i(p_x - ip_y)$. As long as the first supplementary condition on $p_\mu\psi^\mu$ operates onto the Lorentz index only, the latter equations show that it checks consistency with the mass-shell relation $E^2 - \vec{p}^{\,2} = m^2$. For massive particles this condition is fulfilled only for vectors W_1^μ, W_2^μ, and W_3^μ, and not satisfied at all for the vector W_4^μ. This calculation shows that imposing the supplementary condition (7) onto the Rarita-Schwinger field restricts the underlying four vectorial degrees of freedom to only three. However, as shown in Ref. [11], the three vectors W_1^μ, W_2^μ and W_3^μ are not eigenstates of the squared angular momentum $\vec{J}^2$ and do not lend themselves to pure spin-1 states. Rather they are eigenstates of the parity operator, that in the considered representation space is nothing but the matrix of metric tensor $g_{\mu\nu}$=diag$(1,-1,-1,-1)$. Consequently, though this supplementary condition restricts the four degrees of freedom of the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation space to only three, it does not restrict the spin degrees of freedom to spin-1 only.[3]

The spin-0 piece is still there and mixes up with spin-1 within $W_\zeta(\vec{p})$ (for $\zeta = 1,2,3$). The immediate consequence of the covariant inseparability of

[3] Moreover, these three Lorentz vectors cannot span the $\left(\frac{1}{2},\frac{1}{2}\right)$ space in the same mathematical sense as do the four Dirac spinors in the $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$ representation space. It was shown in Ref. [11] that this serious drawback is immediately rectified by incorpoarting, W_4^μ, fourth natural companion of the three W_1^μ, W_2^μ and W_3^μ. Its incorporation into the framework of Rarita and Schwinger, as our experience with the $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$ space suggests, is expected to circumvent the difficulties of quantization pointed out by Weinberg [13].

the $\left(\frac{1}{2},\frac{1}{2}\right)$ space into spin-0 and spin-1 is the covariant inseparability of the Rarita-Schwinger field into a spin-$\frac{3}{2}$ and two spin-$\frac{1}{2}$ components.

Only within the rest frame, or in the helicity basis [11], does the separation between spin-0 and spin-1 take place.

The essential additional physics lies in the fact that the Proca equation

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \tag{19}$$

by construction satisfies, $\partial_\nu A^\nu = 0$ (for $m \neq 0$). However, as shown in Ref. [11], "$\partial_\nu A^\nu = 0$" cannot be satisfied for all relevant degrees of freedom in the massive $\left(\frac{1}{2},\frac{1}{2}\right)$ representation space without violating the completeness relation. While the wave equation satisfied by the $\left(\frac{1}{2},\frac{1}{2}\right)$ spanning W^μ_ζ, contains all solutions of the Proca equation the converse is not true. The wave equation for the W^μ_ζ, which carry with them a completeness relation exactly paralleling the Dirac construct for spin-$\frac{1}{2}$, is [11]:

$$\left(\Lambda_{\mu\nu} p^\mu p^\nu - \epsilon\, m^2 I_4\right) W^\beta_\zeta(\vec{p}) = 0\,, \tag{20}$$

where ϵ equals $+1$ for $\zeta = 4$ and is -1 for $\zeta = 1,2,3$. The $\Lambda_{\mu\nu}$ matrices are:

$$\Lambda_{00} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Lambda_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Lambda_{33} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Lambda_{01} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{02} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_{03} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\Lambda_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{21}$$

The remaining $\Lambda_{\mu\nu}$ are obtained from the above expressions by noting: $\Lambda_{\mu\nu} = \Lambda_{\nu\mu}$.

For this reason, and reasons given in Ref. [11], the Proca equation is not endowed with the complete physical content of the massive $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation space. It is this incompleteness of the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation space when fully removed from the Rarita-Schwinger framework, that one completes the Rarita-Schwinger framework on the one hand, and on the other shows that it calls for a new interpretation and removal of supplementary conditions (6) and (7).

3.3 The Pauli-Lubanski vector

Here we review some necessary kinematic details that may help avoid confusion, and implicitly answer some questions that a reader may raise otherwise.

The Pauli-Lubanski vector is defined as

$$\mathcal{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} I_{\nu\rho} P_\sigma \tag{22}$$

Here, $\epsilon_{\mu\nu\rho\sigma}$ is the standard Levi-Cevita symbol in four dimensions, while $I_{\nu\rho}$ denote the generators of the Lorentz group

$$I_{0i} = K_i\,, \qquad I_{ij} = \epsilon^{ijk} J_k\,, \tag{23}$$

with K_i and J_k being in turn the ith and kth components of boost and rotation generators, respectively. The final expression for the Pauli-Lubanski vector in terms of boost and rotation generators reads for any arbitrary Lorentz group representation

$$\mathcal{W}^\mu = \left[-\vec{J}\cdot\vec{P},\ \ -\vec{J}P_0 + \vec{K}\times\vec{P}\right]\,. \tag{24}$$

Correspondingly, its length, i.e., the second Casimir invariant, is obtained as

$$C_2 \equiv \mathcal{W}^\mu \mathcal{W}_\mu = \left(\vec{J}\cdot\vec{P}\right)^2 - \left(-\vec{J}P_0 + \vec{K}\times\vec{P}\right)^2 \tag{25}$$

It is easy to read off from the latter equation that at rest, C_2 while acting upon a mass eigenstate yields, $-m^2\vec{J}^{\,2}$. Because of that the eigenvalues of the length of the Pauli-Lubanski vector, i.e. C_2, at rest can be given the interpretation of, $-m^2 j(j+1)$. Based upon this finding, valid solely at rest, the impression arises that the second Casimir of the Poincaré group probes both mass and spin of the states. This impression is in general not correct. Indeed, while the eigenvalues of C_2, in being a Casimir operator, are frame independent, their association with the eigenvalues of $\vec{J}^2$ *is frame dependent.* In the most general case the eigenvalues of C_2 arise as a consequence of a delicate cancellation between the actions of all the terms on the right hand side of Eq. (24) upon the state vectors. As a result of these cancellations, even though the eigenvalues of C_2 numerically coincide with the eigenvalues of, $-m^2\vec{J}^{\,2}$, at rest, it has not to be confused with the latter.

Only for the case of single-spin representation sapce the length of the Pauli-Lubanski vector probes both the spin and the mass of the state vectors. And yet, even for the simplest non-trivial single-spin Dirac representation space the relation of the eigenvalue of C_2 to the eigenvalue of $\frac{1}{2}\vec{\sigma}^{\,2}$ in that case, within an arbitrary frame, is not easily recognized. The Pauli-Lubanski vector for the $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$ field reads:

$$\begin{aligned}
\mathcal{W}^\mu &= -\frac{i}{2}\gamma_5\sigma^{\mu\nu}p_\nu \\
&= -\frac{i}{2}\gamma_5\frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)p_\nu \\
&= \frac{1}{4}\gamma_5\left(\gamma^\mu \not{p} - \not{p}\gamma^\mu\right) \\
&= -\frac{1}{4}\gamma_5\left[\not{p},\gamma^\mu\right]_- \, .
\end{aligned} \tag{26}$$

From that the squared of the Pauli-Lubanski vector is easily calculated as

$$\begin{aligned}
C_2 = \mathcal{W}^\mu\mathcal{W}_\mu &= \frac{1}{16}\gamma_5\left[\not{p},\gamma^\mu\right]_-\gamma_5\left[\not{p},\gamma_\mu\right]_- \\
&= \frac{1}{16}\gamma_5^2(\not{p}\gamma^\mu - \gamma^\mu\not{p})(\not{p}\gamma_\mu - \gamma_\mu\not{p}) \\
&= \frac{1}{16}(\not{p}\gamma^\mu\not{p}\gamma_\mu - \not{p}\gamma^\mu\gamma_\mu\not{p} - \gamma^\mu\not{p}\not{p}\gamma_\mu + \gamma^\mu\not{p}\gamma_\mu\not{p}) \\
&= \frac{1}{16}(\not{p}\gamma^\mu\not{p}\gamma_\mu + \gamma^\mu\not{p}\gamma_\mu\not{p} - 4p^2 - 4p^2) \\
&= \frac{1}{16}(\not{p}2p^\mu\gamma_\mu - \not{p}\not{p}\gamma^\mu\gamma_\mu + 2p^\mu\gamma_\mu\not{p} - \not{p}\gamma^\mu\gamma_\mu\not{p} - 8p^2)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{16}(2p^2 - 4p^2 + 2p^2 - 4p^2 - 8p^2) \\
&= \frac{1}{16} 12p^2 \\
&= \frac{3}{4}p^2 \qquad (27)
\end{aligned}
$$

The latter equation shows convincingly that the factor of 3/4 in front of p^2 arises from the general properties of the Clifford algebra of the γ matrices such as anticommutator relations, etc. That the value of this factor is associated with the eigenvalue of $\frac{1}{4}\vec{\sigma}^{\,2}$ is justified only because of the pure-spin character of the $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ state.

We have done detailed calculations and confirmed that only for the $(j,0) \oplus (0,j)$ representation spaces do the single-spin interpretations holds. For representation spaces of the Rarita-Schwinger type even though the action of C_2 on mass eigenstates yields, $-m^2 j(j+1)$, such states, in general, are not eigenstates of $\vec{J}^2$ — where $\vec{J}$ is the appropriate generator of rotation. For the representation space defined by Eq. (1), it is possible, in a rest frame, say, to identify eigenstates of $\vec{J}^2$ as two objects of spin one half, and an object of spin three half. However, this separation into spin-states is not covariant in general — even though the associated C_2 divides the space (1) into sub-spaces that carry eigenvalues, $-m^2\, j(j+1)$, with $j = 1/2, 1/2, 3/2$. The latter eigenvalues of C_2 no longer carry meaning of "spin" in the sense of being eigenstates of $\vec{J}^2$ (which they are not).

4 Conclusion

The main lesson to be learned can now be stated as follows:

(i) Covariance of a set of equations does not guarantee their compatibility with the theory. of relativity.

(ii) Once charge conjugated part of a representation is incorporated in the framework,[4] the eigenvalues of the Casimir invariant C_2 split the representation space into $2(2j+1)$ dimensional subspaces. These subspaces may further subdivide into sectors of definite relative intrinsic parities. In general, however, these subspaces, do not carry a definite spin. That is, they are not eigenstates of the square of the relevant generators of the

[4] Such a charge conjugated sector, e.g., is already present in the $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ part of the Rarita-Schwinger field. For the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation space it can be shown to be brought in by the operation of complex conjugation.

rotations, $\vec{J}^2$. Only in the $(j,0)\oplus(0,j)$ representation spaces do the these subspaces carry definite values of $\vec{J}^2$, C_2, and the parity operator.

We are thus left with no choice but to abandon a single-spin interpretation of the representation space defined in Eq. (1). Once that is done ψ_μ contains two spin one half particles of opposite relative parities, and a spin three half particle. In the absence of any interaction these particles are mass degenerate. Furthermore, the bifurcation into spin one half and spin three half occurs only in the rest and helicity frames. In general, the particles in this representation do not carry a well-defined spin. This interpretation is in accord with the conjecture that one of us advanced some years ago while studying the baryonic spectra [12]. In this representation space the operator $\left(\gamma^\lambda p_\lambda \pm m\right)$ annihilates the spinorial sector of the $\psi_\mu(\vec{p})$, while the operator, $\left(\Lambda_{\lambda\nu}p^\lambda p^\nu \pm m^2 I_4\right)$, annihilates the vector sector of $\psi_\mu(\vec{p})$.

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